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REPRESENTATIONS OF FINITE GROUPS ON RIEMANN-ROCH SPACES, II

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Abstract. If G is a finite subgroup of the automorphism group of a projective curve X and D is a divisor on X stabilized by G, then under the assumption that D is nonspecial, we compute a simplified formula for the trace of the natural representation of G on Riemann-Roch space $L(D)$.

1. Introduction

Let X be a smooth projective (irreducible) curve over an algebraically closed field k and let G be a finite subgroup of automorphisms of X over k. We assume throughout this paper that either char $k = 0$ or char $k = p$ does not divide the order of the group G. If D is a divisor of X which G leaves stable then G acts on the Riemann-Roch space $L(D)$. We are interested in decomposing this representation into irreducibles.

This question was originally addressed by Hurwitz, in the case where D was the canonical divisor and G was cyclic, over $k = \mathbb{C}$. Chevalley and Weil expanded this result to any finite G [CW]. Since then further work has been done by Ellingsrud and Lønsted [EL], Kani [Ka], Nakajima [N], Köck [K1, K2], and Borne [B]. In the case where D is a nonspecial divisor, the character of $L(D)$ has been computed in the work of Borne [B]. We have computed a simpler formula for this character, under a rationality criterion.

Theorem 1. *Let $D = \pi^*(D_0)$ be a nonspecial divisor on X which is a pullback of a divisor D_0 on $Y = X/G$ and assume that the (Brauer) character of $L(D)$ is the character of a $\mathbb{Q}[G]$-module. Then for each absolutely irreducible character of G, the multiplicity of the corresponding module W in $L(D)$ is given by*

$$n = \dim(W)(\deg(D_0) + 1 - g_Y) - \sum_{\ell=1}^{M}(\dim(W) - \dim(W^{H_\ell}))\frac{R_\ell}{2}. \tag{1}$$

The sum is over all conjugacy classes of cyclic subgroups of G, H_ℓ is a representative cyclic subgroup, W^{H_ℓ} indicates the dimension of the fixed part of W under the action of H_ℓ, and R_ℓ denotes the number of branch points in Y where the inertia group is conjugate to H_ℓ.

One motivation for seeking such a formula comes from coding theory. The construction of AG codes uses the Riemann-Roch space $L(D)$ of a divisor on a curve

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defined over a finite field. Automorphisms of $L(D)$ may provide more efficient encoding and storage of information, for some AG codes. See [JT] for more background on AG codes and automorphisms of Riemann-Roch spaces.

In Section 2 we will prove this theorem. In Section 3 we extend to the case that D is not necessarily a pullback. In this case we use a formula due to Borne [B] which expresses $L(D)$ in terms of the equivariant degree of D and the ramification module of the cover, which does not depend on D. Theorem 1 then gives us a simple formula for the ramification module when it obeys the rationality condition. This simple formula for the multiplicity of a $\mathbb{Q}[G]$-module in the ramification module has also been obtained by Köck [K2] using other methods. In Section 4, we give some examples.

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2. Proof of Theorem 1

We start with some definitions and notation.

Let X be a smooth projective (irreducible) curve over an algebraically closed field k and let G be a finite subgroup of automorphisms of X over k. We assume that either char $k = 0$ or char $k = p$ does not divide the order of the group G. For any point $P \in X(k)$, let G_P be the inertia group at P (i.e. the subgroup of G fixing P). Our assumptions on char k ensure that the quotient $\pi : X \to Y = X/G$ is tamely ramified, and this group G_P is cyclic.

Let $\langle G\rangle$ denote the set of conjugacy classes of cyclic subgroups of G. For each class in $\langle G\rangle$ choose a representative cyclic subgroup H_ℓ, $\ell = 1 \dots M$, and partially order them according to the order of the group so that H_1 is the trivial group. For each branch point of the cover $\pi : X \to Y$, the inertia groups at the ramification points P above that branch point will be cyclic and conjugate to each other. For each ℓ, let R_ℓ denote the number of branch points in Y where the inertia groups are conjugate to H_ℓ. (R_1 may be set to 0; it does not play a role in the formula).

Let $G^*_{\mathbb{Q}}$ denote the set of equivalence classes of irreducible $\mathbb{Q}[G]$-modules. By results in ([Se], §13.1, §12.4), this set has the same number of elements, M, as $\langle G\rangle$. For each class in $G^*_{\mathbb{Q}}$, choose a representative irreducible $\mathbb{Q}[G]$-module V_j, $j = 1 \dots M$, and denote its character by χ_j. The character table of G over $\mathbb{Q}$ is a square matrix with rows labelled by $G^*_{\mathbb{Q}}$ and columns labelled by $\langle G\rangle$. The rows are linearly independent (as $\mathbb{Q}$-class functions), so in fact the character table is an invertible matrix.

Let F be a finite extension of $\mathbb{Q}$ such that every irreducible $F[G]$-module is absolutely irreducible (irreducible over $\mathbb{C}$), so that the character table of G over F is the same as the character table for G over $\mathbb{C}$ ([Se], p. 94). For each irreducible $\mathbb{Q}[G]$-module V_j, $V_j \otimes_{\mathbb{Q}[G]} F[G]$ decomposes into irreducible $F[G]$-modules. The Galois group of F over $\mathbb{Q}$ permutes the components transitively, so each must have the same multiplicity (the Schur index of the representation V_j) and the same dimension. We write

$$V_j \otimes_{\mathbb{Q}[G]} F[G] \simeq m_j \bigoplus_{r=1}^{d_j} W_{jr}, \tag{2}$$

where m_j is the Schur index, the W_{jr}'s are irreducible $F[G]$-modules, and $\dim_{\mathbb{Q}} V_j = m_j d_j \dim_F W_{jr}$ for each r. Let χ_{jr} denote the character of W_{jr}.

Theorem 1 is a consequence of the following.

Theorem 2. *Let $D = \pi^*(D_0)$ be a nonspecial divisor on X and assume that the (Brauer) character of $L(D)$ is the character of a $\mathbb{Q}[G]$-module $L(D)_{\mathbb{Q}}$. Then for each irreducible $\mathbb{Q}[G]$-module V_j, its multiplicity in $L(D)_{\mathbb{Q}}$ is given by*

$$n_j = \frac{1}{m_j^2 d_j}\left(\dim(V_j)(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(V_j)-\dim(V_j^{H_\ell}))\frac{R_\ell}{2}\right). \tag{3}$$

Proof: The proof is similar to the proof of Theorem 2.3 in [Ks]. We consider the quotients X/H_ℓ of X by cyclic subgroups H_ℓ. The morphism $\pi : X \to Y$ factors through this quotient, so on each X/H_ℓ there is a pullback divisor D_ℓ of D_0.

First, note that our assumption that D is nonspecial means that for any quotient X/H_ℓ, the pullback D_ℓ of D_0 to X/H_ℓ is also nonspecial. This is because

$$K_X - D = \pi_\ell^*(K_{X/H_\ell}) + Ram(X/H_\ell) - \pi_\ell^*(D_\ell) = \pi_\ell^*(K_{X/H_\ell} - D_\ell) + Ram(X/H_\ell)$$

where $Ram(X/H_\ell)$ is the ramification divisor of the covering $\pi_\ell : X \to X/H_\ell$. Any element of $L(K_{X/H_\ell} - D_\ell)$ would pull back to X to give an element of $L(K_X - D - Ram(X/H_\ell))$. Since $Ram(X/H_\ell)$ is effective, this would also give an element of $L(K_X - D)$, contradicting our assumption that D is nonspecial.

Now we decompose $L(D)_{\mathbb{Q}}$ as

$$L(D)_{\mathbb{Q}} \simeq \bigoplus_{j=1}^{M} n_j V_j. \tag{4}$$

For each H_ℓ in $\langle G\rangle$, consider the dimension of the piece of this module fixed by H_ℓ. Since the elements of $L(D)$ fixed by H_ℓ are exactly the elements of $L(D_\ell)$, $\dim_{\mathbb{Q}} L(D)_{\mathbb{Q}}^{H_\ell} = \dim_k L(D)^{H_\ell} = \dim_k L(D_\ell)$ and we get an equation for each ℓ:

$$\dim_k L(D_\ell) = \sum_{j=1}^{M} n_j \dim_{\mathbb{Q}}(V_j^{H_\ell}), \qquad 1 \le \ell \le M. \tag{5}$$

This gives us a system of M equations in the M unknowns n_j. We need to show that the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is invertible, so this system has a unique solution, and that the above equation is the claimed solution.

First let us consider the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$. Each matrix entry is equal to the multiplicity of the trivial representation of H_ℓ in the restricted representation of H_ℓ on V_j. This is the inner product of characters $\langle \mathrm{Res}_{H_\ell}^G \chi_j, \mathbf{1}\rangle$, which is defined as

$$\dim V_j^{H_\ell} = \frac{1}{|H_\ell|}\sum_{a\in H_\ell}\chi_j(a) \tag{6}$$

Thus each column of the matrix $(\dim(V_j^{H_\ell}))_{j,\ell}$ is a sum of columns of the character table of G over $\mathbb{Q}$. Each element a in H_ℓ generates either all of H_ℓ or a cyclic subgroup of lower order, hence earlier in the list $\langle G\rangle$. Thus if we write our matrix in terms of the basis of columns of this character table, we get a lower triangular

matrix with nonzero entries on the diagonal. This implies that our matrix is also invertible.

Now it remains to verify that our equation is the correct solution to (5).

Note that

$$\dim L(D_\ell) = \frac{|G|}{|H|}\deg(D_0) + 1 - g(X/H_l), \tag{7}$$

for $1 \le \ell \le M$, by the Riemann-Roch theorem and the hypothesis that D_ℓ is nonspecial.

We will now substitute (3) into (5) and verify that the result agrees with (7), for each $1 \le \ell \le M$. The argument is similar to that in [Ks].

Plugging (3) into (5) gives

$$\begin{aligned}\sum_{j=1}^{M} n_j \dim(V_j^{H_\ell}) &= (\deg(D_0) + 1 - g_Y)\sum_{j=1}^{M} \frac{1}{m_j^2 d_j}\dim(V_j^{H_\ell})\dim(V_j) \\ &\quad - \sum_{i=1}^{M}\left(\sum_{j=1}^{M} \frac{1}{m_j^2 d_j}[\dim(V_j^{H_\ell})\dim(V_j) - \dim(V_j^{H_\ell})\dim(V_j^{H_i})]\frac{R_i}{2}\right)\end{aligned}$$

Note that

$$\dim(V_j^{H_\ell}) = \langle \mathrm{Res}_{H_\ell}^G \chi_j, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle = m_j \sum_{r=1}^{d_j} \langle \chi_{jr}, \mathrm{Ind}_{H_\ell}^G \mathbf{1}\rangle, \tag{8}$$

using (2) and Frobenius reciprocity. This gives us

$$\begin{aligned}\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim V_j^{H_\ell} \dim V_j &= \sum_{j=1}^{M} \frac{\dim V_j}{m_j d_j} \sum_{r=1}^{d_j} \langle \chi_{jr}, \mathrm{Ind}_{H_\ell}^G \mathbf{1}\rangle \\ &= \sum_{j=1}^{M}\sum_{r=1}^{d_j} \dim W_{jr} \langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle \\ &= \frac{1}{|H_\ell|} \sum_{a \in H_\ell} \sum_{j=1}^{M}\sum_{r=1}^{d_j} \chi_{jr}(e)\chi_{jr}(a)\end{aligned} \tag{9}$$

The last part of this is summing over all irreducible F-characters of G, so the last expression is in fact the inner product of two columns of the character table for G over F. This inner product will be zero unless $a = e$, so the sum becomes

$$\frac{1}{|H_\ell|}\sum_{j=1}^{M}\sum_{r=1}^{d_j} \chi_{jr}(e)^2 = \frac{|G|}{|H_\ell|}. \tag{10}$$

We would like to do a similar simplification of

$$\sum_{j=1}^{M} \frac{1}{m_j^2 d_j} \dim(V_j^{H_\ell}) \dim(V_j^{H_i}) \tag{11}$$

using (8) twice. The induced representation $\mathrm{Ind}_{H_i}^G \mathbf{1}$ is the action of G by permutations on the cosets of H_i, and thus has a $\mathbb{Q}[G]$-module structure as well as an $F[G]$-module structure. It can be decomposed into irreducible $F[G]$-modules, such that for each j the multiplicities of the W_{jr}'s, $\langle \chi_{jr}, \mathrm{Ind}_{H_i}^G \mathbf{1}\rangle$, are all equal. Using that fact, Frobenius reciprocity, and the definition of the Schur inner product, we have

$$
\begin{aligned}
\textstyle\sum_{j=1}^M \frac{1}{m_j^2 d_j} \quad & \dim(V_j^{H_\ell})\dim(V_j^{H_i}) \\
&= \textstyle\sum_{j=1}^M \frac{1}{d_j}\sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle \sum_{s=1}^{d_j}\langle \chi_{js}, \mathrm{Ind}_{H_i}^G \mathbf{1}\rangle \\
&= \textstyle\sum_{j=1}^M \sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle\langle \chi_{jr}, \mathrm{Ind}_{H_i}^G \mathbf{1}\rangle \\
&= \textstyle\sum_{j=1}^M \sum_{r=1}^{d_j}\langle \mathrm{Res}_{H_\ell}^G \chi_{jr}, \mathbf{1}\rangle\langle \mathrm{Res}_{H_i}^G \chi_{jr}, \mathbf{1}\rangle \\
&= \textstyle\frac{1}{|H_\ell|}\frac{1}{|H_i|}\sum_{a\in H_\ell}\sum_{b\in H_i}\sum_{j=1}^M\sum_{r=1}^{d_j}\chi_{jr}(a)\chi_{jr}(b).
\end{aligned} \tag{12}
$$

Again, this last is an inner product of columns of the character table of G over k, so will be zero unless a and b are in the same conjugacy class. Let $\mathcal{C}_G(a)$ denote the conjugacy class of a in G. We end up with

$$
\begin{aligned}
\sum_{j=1}^M \frac{1}{m_j^2 d_j}\dim(V_j^{H_\ell})\dim(V_j^{H_i}) &= \frac{1}{|H_\ell||H_i|}\sum_{a\in H_\ell}\#(H_i\cap\mathcal{C}_G(a))\sum_{j=1}^M\sum_{r=1}^{d_j}\chi_{jr}(a)^2 \\
&= |H_\ell\backslash G/H_i|
\end{aligned} \tag{13}
$$

the number of double cosets.

From this we get

$$
\begin{aligned}
\sum_{j=1}^M n_j \dim V_j^{H_\ell} &= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} - \sum_{i=1}^M\left(\frac{|G|}{|H_\ell|} - |H_i\backslash G/H_\ell|\right)\frac{R_i}{2} \\
&= (\deg(D_0)+1-g_Y)\frac{|G|}{|H_\ell|} + 1 + \frac{|G|}{|H_\ell|}(g_Y-1) - g_{X/H_\ell} \\
&= \deg(D_0)\frac{|G|}{|H_\ell|} + 1 - g_{X/H_\ell}.
\end{aligned}
$$

where the last equalities come from applying the Hurwitz formula to the cover $X/H_\ell \to Y$ (see [Ks] for details). This is (7), as desired. □

Proof of Theorem 1. We use the decomposition (2) to compute the multiplicity of each W_{jr} in $L(D)_{\mathbb{Q}}\otimes F$. By our definition of F, each absolutely irreducible character is the character of one of the W_{jr}'s, and the character of $L(D)$ is the same as the character of $L(D)_{\mathbb{Q}}\otimes F$, so this will give us the correct answer.

The multiplicity of W_{jr} in V_j is m_j, and $\dim V_j = m_j d_j \dim W_{jr}$. Equation 8 and the fact that $\mathrm{Ind}_{H_\ell}^G \mathbf{1}$ has a $\mathbb{Q}[G]$-module structure means that $\dim W_{jr}^{H_\ell}$ is the same for each r, so $\dim V_j^{H_\ell} = m_j d_j \dim W_{jr}^{H_\ell}$. Thus we can factor $m_j d_j$ out from the inside and multiply the whole thing by m_j to get formula (1). □

Remark. The rationality criterion is necessary for this formula to be accurate. If the character of $L(D)$ is not the character of a $\mathbb{Q}[G]$-module, it will still be the character of an F-module $L(D)_F$, and $L(D)_F$ will decompose into irreducibles W_{jr}. However in this case for each j, the multiplicities of the W_{jr}'s may not be all the

same. The right hand side of equation (1) will then compute the average of these multiplicities:

$$\frac{1}{d_j}\sum_{r=1}^{d_j}\langle\chi_{jr}, L(D)\rangle = \dim(W_{jr})(\deg(D_0)+1-g_Y) - \sum_{\ell=1}^{M}(\dim(W_{jr}) - \dim(W_{jr}^{H_\ell}))\frac{R_\ell}{2}. \tag{14}$$

3. D IS NOT A PULLBACK

Now we wish to extend our results to the case where D is not necessarily the pullback of a divisor on $Y = X/G$. For this we need to build on work previously done on this problem by Nakajima, Borne, Ellingsrud and Lønsted, Köck, Kani, and others. We refer to [B] for references. We start with two definitions: the ramification module of the cover $X \to X/G$ and the equivariant degree of a divisor.

For any point $P \in X(k)$, the inertia group G_P acts on the cotangent space of $X(k)$ at P by a k-character ψ_P. This character is the **ramification character** of X at P. The ramification module is defined by

$$\Gamma_G = \sum_{P\in X(k)_{ram}} \mathrm{Ind}_{G_P}^G(\sum_{\ell=1}^{e_P-1}\ell\psi_P^\ell),$$

where $e_P = |G_P|$. By Theorem 2 in [N], there is a unique G-module $\tilde{\Gamma}_G$ such that

$$\Gamma_G = |G|\tilde{\Gamma}_G.$$

In this paper we are only concerned with $\tilde{\Gamma}_G$, so we abuse terminology and call $\tilde{\Gamma}_G$ the **ramification module**.

Now consider a G-invariant divisor D on $X(k)$. If $D = \frac{1}{e_P}\sum_{g\in G} g(P)$ then we call D a **reduced orbit**. The reduced orbits generate the group of G-invariant divisors $Div(X)^G$.

Definition 3. *The* ***equivariant degree*** *is a map from $Div(X)^G$ to the Grothendieck group $R_k(G) = \mathbb{Z}[G_k^*]$ of virtual k-characters of G,*

$$deg_{eq} : Div(X)^G \to R(G),$$

defined by the following conditions:

1. *deg_{eq} is additive on G-invariant divisors of disjoint support,*
2. *If $D = r\frac{1}{e_P}\sum_{g\in G} g(P)$ is an orbit then*

$$deg_{eq}(D) = \begin{cases} Ind_{G_P}^G(\sum_{\ell=1}^{r}\psi_P^{-\ell}), & \text{if } r > 0, \\ -Ind_{G_P}^G(\sum_{\ell=0}^{-(r+1)}\psi_P^{\ell}), & \text{if } r < 0, \\ 0, & r = 0, \end{cases}$$

where ψ_P is the ramification character of X at P.

Lemma 4. (Borne's formula) *If D is a G-equivariant nonspecial divisor, then the (virtual) character of $L(D)$ is given by*

$$\chi(L(D)) = (1-g_Y)\chi(k[G]) + deg_{eq}(D) - \chi(\tilde{\Gamma}_G). \tag{15}$$

We derive the following from Borne's formula and Theorem 1. The notation is as in Section 1.

Proposition 5. *If $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure, then it decomposes into irreducible $\mathbb{Q}[G]$-modules as*

$$\tilde{\Gamma}_G \simeq \bigoplus_j \frac{1}{m_j^2 d_j}(\sum_\ell (\dim(V_j) - \dim(V_j^{H_\ell}))\frac{R_\ell}{2})V_j.$$

Proof: The ramification module does not depend on the divisor, so we compare Theorem 1 with Borne's formula in the case where D is a pullback. If $D = \pi^*(D_0)$ is the pull-back of a divisor $D_0 \in Div(Y)$ then the equivariant degree $deg_{eq}(D)$ has a very simple form. On each orbit, r is a multiple of e_P, so every character of the cyclic group G_P appears. The equivariant degree on this orbit is induced from a multiple of the regular representation of G_P. Thus we have

$$deg_{eq}(D) = deg(D_0)\chi(k[G]), \tag{16}$$

(This is also a special case of Corollary 3.10 in [B].)

The first two terms of Borne's formula then become

$$(\deg D_0 + 1 - g_Y)\chi(k[G]).$$

This is clearly the character of a $\mathbb{Q}[G]$-module, so $L(D)$ will have a $\mathbb{Q}[G]$-module structure if and only if $\tilde{\Gamma}_G$ does. The rest of the proposition follows from Theorem 1. □

Proposition 5 has also been proven by Köck [K2], using a different method.

Corollary 6. *Suppose that $\tilde{\Gamma}_G$ has a $\mathbb{Q}[G]$-module structure. Let W be an irreducible $F[G]$-module. Then the multiplicity of the character of W in $\tilde{\Gamma}_G$ is*

$$\sum_\ell (\dim(W) - \dim(W^{H_\ell}))\frac{R_\ell}{2}. \tag{17}$$

Proof: The same as the proof of Theorem 1 from Theorem 2. □

Remark. Again, the rationality criterion is necessary. If $\tilde{\Gamma}_G$ does not have a $\mathbb{Q}[G]$-module structure, we get an average of multiplicities, similar to (14):

$$\frac{1}{d_j}\sum_{r=1}^{d_j}\langle \chi_{jr}, \tilde{\Gamma}_G\rangle = \sum_{\ell=1}^{M}(\dim(W_{jr}) - \dim(W_{jr}^{H_\ell}))\frac{R_\ell}{2} \tag{18}$$

with notation is as in (2).

4. Examples

Example 1. Consider the nonsingular projective curve X which is the closure of

$$\{(x, y, t) \in \mathbb{C}^3 \mid y^2 = x(x-2)(x-4),\ t^2 = x + 4\}.$$

This has an action of $G = C_2 \times C_2$ given by

$$\begin{aligned} \alpha : (x, y, z) &\longmapsto (x, -y, t), \\ \beta : (x, y, z) &\longmapsto (x, y, -t), \\ \alpha\beta : (x, y, z) &\longmapsto (x, -y, -t). \end{aligned}$$

The quotient by β is a degree two cover of an elliptic curve, ramified at the two points with $x = -4$, so X has genus 2. The quotient $Y = X/G$ is the projective x-line.

The divisor

$$D = (0, 0, 2) + (0, 0, -2) + (-4, 8\sqrt{3}, 0) + (-4, -8\sqrt{3}, 0)$$

is G-equivariant, and $2D$ is the pullback of the divisor $D_0 = x = 0, x = -4$ on Y. From the Riemann-Roch theorem we know that $\dim L(2D) = 7$.

First, let us use Theorem 1 to decompose $L(2D)$ into irreducibles. The cyclic subgroups of G are the trivial group, H_1 and each of the two-element subgroups generated by α, β, and $\alpha\beta$. Let us call the last three H_α, H_β, and $H_{\alpha\beta}$. Each is in its own conjugacy class.

The cover $X \to X/G$ has 5 branch points: three with inertia group H_α (at $x = 0, 2, 4$), one with inertia group H_β (at $x = -4$), and one with inertia group $H_{\alpha\beta}$ (at $x = \infty$). This means

$$R_\alpha = 3, \quad R_\beta = 1, \quad R_{\alpha\beta} = 1.$$

The group G has character table

	1	α	β	$\alpha\beta$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Each irreducible representation is one dimensional, and every $\mathbb{C}[G]$-module is a $\mathbb{Q}[G]$-module, so d_j and the Schur index m_j are both 1. The dimension $\dim(V_j^{H_\ell})$ is 1 if the character of V_j is 1 on the generator and 0 otherwise. From this we get:

$$\begin{aligned}
n_1 &= (2+1-0) - 0 = 3 \\
n_2 &= (2+1-0) - \frac{1}{2}(R_\beta + R_{\alpha\beta}) = 3 - 1 = 2 \\
n_3 &= (2+1+0) - \frac{1}{2}(R_\alpha + R_{\alpha\beta}) = 3 - 2 = 1 \\
n_4 &= (2+1+0) - \frac{1}{2}(R_\alpha + R_\beta) = 3 - 2 = 1.
\end{aligned}$$

Thus the character of $L(2D)$ is $3\chi_1 + 2\chi_2 + \chi_3 + \chi_4$.

Now let us consider $L(D)$. The Riemann-Roch theorem tells that that this will be a three dimensional space. Since D is not a pullback from Y, we cannot use Theorem 1. However, the ramification module does have a $\mathbb{Q}[G]$-module structure, so we can use Proposition 5 with Borne's formula. The calculations above tell us that the ramification module has character $\chi_2 + 2\chi_3 + 2\chi_4$.

Now we need to calculate the equivariant degree of D. The divisor consists of two reduced orbits, the orbit of $(0, 0, 2)$ and the orbit of $(-4, 8\sqrt{3}, 0)$. At the first point the inertia group is H_α, and at the second point the inertia group is H_β. In both cases the ramification character is the nontrivial character of C_2. Adding the induced characters of G gives us $\deg_{eq}(D) = \chi_2 + \chi_3 + 2\chi_4$.

Adding the pieces of Borne's formula, we get the character of $L(D)$ to be $\chi_1 + \chi_2 + \chi_4$. In fact, one can check that the three functions $\{1, \frac{1}{t}, \frac{y}{xt}\}$ form a basis for $L(D)$, and G acts on the three basis elements by the three respective characters. $\square$

Example 2. Let $X = \mathbb{P}^1$ and let G be a cyclic group of prime order q. Let a be a generator of G, and let a act on X by $z \longmapsto \zeta z$, where ζ is a primitive qth root of unity. The cyclic subgroups of G are the trivial group and G itself; the irreducible representations of G over $\mathbb{Q}$ are the one-dimensional trivial representation and a $q-1$ dimensional representation V. Let ψ be the character of G over $\mathbb{C}$ whose value on a is ζ; then the irreducible characters of G over $\mathbb{C}$ are the tensor powers $\psi, \psi^{\otimes 2}, \dots, \psi^{\otimes q-1}, \psi^{\otimes q} = \mathbf{1}$. The character of V is $\psi + \psi^2 + \dots + \psi^{q-1}$.

The cover $X \to X/G$ is totally ramified at 0 and ∞. The ramification module in this case is a $\mathbb{Q}[G]$-module, so we can use either Proposition 5 or Corollary 6 to find that

$$\tilde{\Gamma}_G = \psi + \psi^2 + \dots \psi^{q-1} = V. \quad \square$$

The following example illustrates what can happen when the rationality condition is not met.

Example 3. Let X be the Klein quartic

$$\{\ (x, y, z) \in \mathbb{P}^2 \mid x^3y + y^3z + z^3x = 0\ \}.$$

We assume that k contains both cube roots of unity and 7^{th} roots of unity; let ω be a primitive cube root of unity and ζ be a primitive seventh root of unity. Let G be the group generated by

$$\begin{array}{rcl} \sigma : (x:y:z) & \longmapsto & (y:z:x) \\ \tau : (x:y:z) & \longmapsto & (\zeta x : \zeta^4 y : \zeta^2 z) \end{array}$$

The group G of automorphisms generated by these two actions is the semi-direct product $C_3 \rtimes C_7$. (This is not the full automorphism group of this curve.) X has genus 2, and the quotient $Y = X/G$ has genus 0 [E].

The group G has character table[1]:

	e	σ	τ	σ^{-1}	τ^{-1}
χ_1	1	1	1	1	1
χ_2	1	ω^2	1	ω	1
χ_3	1	ω	1	ω^2	1
χ_4	3	0	$\zeta^3+\zeta^5+\zeta^6$	0	$\zeta+\zeta^2+\zeta^4$
χ_5	3	0	$\zeta+\zeta^2+\zeta^4$	0	$\zeta^3+\zeta^5+\zeta^6$

There are two conjugacy classes of nontrivial cyclic subgroups, with representatives generated by σ and τ. Let $H_3 = \langle\sigma\rangle$ and $H_7 = \langle\tau\rangle$. The irreducible representations over $\mathbb{Q}$ have characters χ_1, $\chi_2 + \chi_3$, and $\chi_4 + \chi_5$. Each has Schur index 1.

The points of X fixed by H_7 are $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, and $P_3 = (0,0,1)$. These form one orbit under G, so $R_7 = 1$. There are seven points in the orbit of $(1:\omega:\omega^2)$ and seven points in the orbit of $(1:\omega^2:\omega)$, all fixed by cyclic groups of order 3. Since these form two orbits, we have $R_3 = 2$.

[1]This was obtained using [Gap]. Incidentally, there is only one non-cyclic group of order 21, up to isomorphism.

We now compute

$$\sum_{\ell=1}^{M}(\dim(W)-\dim(W^{H_\ell}))\frac{R_\ell}{2}, \tag{19}$$

as in (18), for the irreducible representations over $\mathbb{C}$. We find that

$$\begin{aligned}\frac{1}{2}\,\langle \chi_2+\chi_3,\tilde{\Gamma}_G\rangle &= 1\\ \frac{1}{2}\,\langle \chi_4+\chi_5,\tilde{\Gamma}_G\rangle &= \frac{7}{2}\end{aligned}$$

These give the average multiplicities. In fact one can compute directly that $\tilde{\Gamma}_G = \chi_2+\chi_3+3\chi_4+4\chi_5$.

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